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Cauchy problem for the complex Ginzburg-Landau equation with harmonic oscillator

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1. Introduction and results

Let $N \in \mathbb{N}$. This paper is concerned with the following Cauchy problem for the complex Ginzburg-Landau equation with *Laplacian replaced with Hamiltonian for harmonic oscillator*:

$$(\text{CGL})_{\mathbb{R}^N, \mu} \quad \begin{cases} \frac{\partial u}{\partial t} + (\lambda + i\alpha)(-\Delta + \mu^2|x|^2)u + (\kappa + i\beta)|u|^{q-2}u - \gamma u = 0 & \text{on } \mathbb{R}^N \times \mathbb{R}_+, \\ u(x, 0) = u_0(x), \quad x \in \mathbb{R}^N, \end{cases}$$

where $\lambda, \kappa \in \mathbb{R}_+ := (0, \infty)$, $\alpha, \beta, \gamma \in \mathbb{R}$, $\mu > 0$ and $q \geq 2$ are constants, and $u = u(x, t)$ is a complex-valued unknown function. In particular, the case where $\mu = 0$, i.e., $(\text{CGL})_{\mathbb{R}^N, 0}$ is a Cauchy problem for the *usual* complex Ginzburg-Landau equation which is also regarded as the special case of initial-boundary value problem of the form

$$(\text{CGL})_{\Omega, 0} \quad \begin{cases} \frac{\partial u}{\partial t} - (\lambda + i\alpha)\Delta u + (\kappa + i\beta)|u|^{q-2}u - \gamma u = 0 & \text{on } \Omega \times \mathbb{R}_+, \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}_+, \\ u(x, 0) = u_0(x), \quad x \in \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a general domain with boundary $\partial\Omega$. For physical background of the complex Ginzburg-Landau equation see e.g., Aranson-Kramer [1].

The purpose of this paper is to discuss the following three problems.

(Problem 1) Existence of global strong solutions to $(\text{CGL})_{\mathbb{R}^N, \mu}$.

(Problem 2) Uniqueness of global strong solutions to $(\text{CGL})_{\mathbb{R}^N, \mu}$.

(Problem 3) Existence of global strong solutions to $(\text{CGL})_{\mathbb{R}^N, 0}$ by letting $\mu \downarrow 0$ in $(\text{CGL})_{\mathbb{R}^N, \mu}$.

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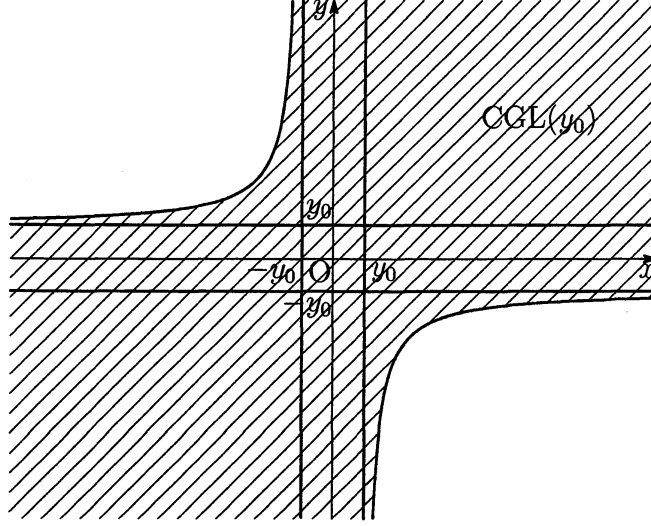


Figure 1: The boundary of $CGL(y_0)$ is given by a pair of hyperbolas.

To clarify the problem we review the known results. Ginibre-Velo [2] established the existence (except uniqueness) of global strong solutions to $(CGL)_{\mathbb{R}^N,0}$ with $u_0 \in H^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ under the condition that

$$(1.1) \quad \left(\frac{\alpha}{\lambda}, \frac{\beta}{\kappa} \right) \in CGL(c_q^{-1}) := \left\{ (x, y) \in \mathbb{R}^2; xy \geq 0 \text{ or } \frac{|xy| - 1}{|x| + |y|} < \frac{1}{c_q} \right\},$$

$$(1.2) \quad c_q := \frac{q-2}{2\sqrt{q-1}}$$

(see Figure 1). Condition (1.1) plays an essential role in deriving the estimates of

$$\begin{aligned} & (\delta^2/2) \|\nabla u(t)\|_{L^2}^2 + (1/q) \|u(t)\|_{L^q}^q, \\ & \int_0^t \{ \delta^2 \|\Delta u(s)\|_{L^2}^2 + \|u(s)\|_{L^{2(q-1)}}^{2(q-1)} \} ds \end{aligned}$$

for some $\delta > 0$. In [2, Proof of Proposition 5.1] they used compactness methods; however, their proof is much complicated since both the nonlinear term and the initial data are regularized. The result is extended to problem $(CGL)_{\Omega,0}$ in a *bounded* domain Ω (see Okazawa-Yokota [5, Theorem 1.1 with $p = 2$]). However, when Ω is an *unbounded* general domain and $q \geq 2$ is not restricted by N , there seems to be no work except the case where

$$\begin{aligned} & \left(\frac{\alpha}{\lambda}, \frac{\beta}{\kappa} \right) \in S(c_q^{-1}) := \left\{ (x, y) \in \mathbb{R}^2; |y| \leq \frac{1}{c_q} \right\} \subset CGL(c_q^{-1}), \\ & \left(\iff \frac{|\beta|}{\kappa} \leq \frac{1}{c_q} \right). \end{aligned}$$

This implies that the mapping $u \mapsto -(\lambda + i\alpha)\Delta u + (\kappa + i\beta)|u|^{q-2}u$ is accretive in $L^2(\Omega)$. In this case the existence and uniqueness of global strong solutions to $(CGL)_{\Omega,0}$ with

$u_0 \in L^2(\Omega)$ are obtained in [5, Theorem 1.3 with $p = 2$]. Therefore the problem lies in the case where Ω is unbounded and $(\alpha/\lambda, \beta/\kappa) \in CGL(c_q^{-1}) \setminus S(c_q^{-1})$. In this paper we give a partial answer to the case where $\Omega = \mathbb{R}^N$ via compactness methods by adding the harmonic oscillator $|x|^2$.

Before stating our results, we define a global strong solution to $(CGL)_{\mathbb{R}^N, \mu}$.

Definition 1.1. A function $u(\cdot) \in C([0, \infty); L^2(\mathbb{R}^N))$ is said to be a *global strong solution* to $(CGL)_{\mathbb{R}^N, \mu}$ if $u(\cdot)$ has the following properties:

- (a) $u(t) \in H^2(\mathbb{R}^N) \cap L^{2(q-1)}(\mathbb{R}^N)$, $|x|^2 u(t) \in L^2(\mathbb{R}^N)$ a.a. $t > 0$;
- (b) $(\partial u / \partial t)(\cdot)$, $\Delta u(\cdot)$, $|x|^2 u(\cdot)$, $|u|^{q-2} u(\cdot) \in L^2(0, T; L^2(\mathbb{R}^N))$ for every $T > 0$;
- (c) $u(\cdot)$ satisfies the equation in $(CGL)_{\mathbb{R}^N, \mu}$ a.e. on \mathbb{R}_+ as well as the initial condition.

First we give an answer to **Problem 1**. Using the compactness of $(-\Delta + \mu^2 |x|^2)^{-1}$ ($\mu > 0$) in $L^2(\mathbb{R}^N)$ (see Okazawa [4]), we can establish the existence of global strong solutions to $(CGL)_{\mathbb{R}^N, \mu}$ with $u_0 \in H^1(\mathbb{R}^N) \cap D(|x|) \cap L^q(\mathbb{R}^N)$ under condition (1.1). Here $D(|x|)$ is regarded as a Hilbert space given by

$$\begin{aligned} D(|x|) &:= \{u \in L^2(\mathbb{R}^N); |x|u \in L^2(\mathbb{R}^N)\}, \\ (u, v)_{D(|x|)} &:= (u, v)_{L^2} + (|x|u, |x|v)_{L^2}, \quad u, v \in D(|x|). \end{aligned}$$

Theorem 1.1. Let $N \in \mathbb{N}$, $\lambda > 0$, $\kappa > 0$, $\alpha, \beta, \gamma \in \mathbb{R}$ and $\mu > 0$. Assume that condition (1.1) is satisfied. Then for any $u_0 \in H^1(\mathbb{R}^N) \cap D(|x|) \cap L^q(\mathbb{R}^N)$ there exists a global strong solution $u(\cdot) \in C([0, \infty); L^2(\mathbb{R}^N))$ to $(CGL)_{\mathbb{R}^N, \mu}$ such that

$$(1.3) \quad u(\cdot) \in C([0, \infty); H^1(\mathbb{R}^N) \cap D(|x|) \cap L^q(\mathbb{R}^N)),$$

with the estimates for every $t > 0$

$$(1.4) \quad \|u(t)\|_{L^2} \leq e^{\gamma t} \|u_0\|_{L^2},$$

$$(1.5) \quad E_\mu(u(t)) + \eta \int_0^t \{\delta^2 \|(\Delta - \mu^2 |x|^2)u(s)\|_{L^2}^2 + \|u(s)\|_{L^{2(q-1)}}^{2(q-1)}\} ds \leq e^{\gamma + \eta t} E_\mu(u_0),$$

where

$$E_\mu(u) := \frac{\delta^2}{2} [\|\nabla u\|_{L^2}^2 + \mu^2 \| |x|u \|_{L^2}^2] + \frac{1}{q} \|u\|_{L^q}^q,$$

$\gamma_+ := \max\{\gamma, 0\}$ and $\delta > 0$, $\eta > 0$ are constants depending only on $\lambda, \kappa, \alpha, \beta, q$.

Secondly we give an answer to **Problem 2** under the additional condition

$$(1.6) \quad 2 \leq q < 2^* := \begin{cases} 2 + \frac{4}{N-2} & (N \geq 3), \\ \infty & (N = 1, 2). \end{cases}$$

This condition appeared in proving the uniqueness of solutions to $(CGL)_{\mathbb{R}^N, 0}$ or $(CGL)_{\Omega, 0}$ (see Ginibre-Velo [3, Proposition 4.2] and Okazawa-Yokota [6, Theorem 1.2]).

Theorem 1.2. Let $N \in \mathbb{N}$, $\lambda > 0$, $\kappa > 0$, $\alpha, \beta, \gamma \in \mathbb{R}$ and $\mu > 0$. Assume that (1.1) and (1.6) are satisfied. Then the solutions to $(\text{CGL})_{\mathbb{R}^N, \mu}$ in the sense of Definition 1.1 are unique. In fact, let $u(\cdot)$ and $v(\cdot)$ be global strong solutions to $(\text{CGL})_{\mathbb{R}^N, \mu}$ with initial data $u_0, v_0 \in H^1(\mathbb{R}^N) \cap D(|x|)$, respectively. Set $w(\cdot) := u(\cdot) - v(\cdot)$ and $w_0 := u_0 - v_0$. Then

$$(1.7) \quad \|w(t)\|_{L^2}^2 + \lambda \int_0^t e^{\int_s^t K(r) dr} \{ \|\nabla w(s)\|_{L^2}^2 + \mu^2 \| |x| w(s) \|_{L^2}^2 \} ds \leq e^{\int_0^t K(r) dr} \|w_0\|_{L^2}^2, \quad t > 0,$$

where $K(\cdot)$ is a continuous function depending only on $\lambda, \kappa, \beta, \gamma, q, E_\mu(u_0)$ and $E_\mu(v_0)$.

Finally, combining Theorems 1.1 and 1.2, we can give an answer to **Problem 3** under (1.6). The following theorem is the special case of [2, Proposition 5.1] concerning the existence; however, our approach here is *much simpler* than that in [2].

Theorem 1.3. Let $N \in \mathbb{N}$, $\lambda > 0$, $\kappa > 0$, $\alpha, \beta, \gamma \in \mathbb{R}$ and $\mu > 0$. Assume that conditions (1.1) and (1.6) are satisfied. Let $\{u_\mu(\cdot)\}_{\mu>0}$ be a family of unique global strong solutions to $(\text{CGL})_{\mathbb{R}^N, \mu}$ with initial data $u_0 \in H^1(\mathbb{R}^N) \cap D(|x|^2)$. Then

$$u(\cdot) := \lim_{\mu \downarrow 0} u_\mu(\cdot)$$

gives a (unique) global strong solution to $(\text{CGL})_{\mathbb{R}^N, 0}$ with $u(0) = u_0$.

The proofs of Theorems 1.1, 1.2 and 1.3 are given in Sections 2, 3 and 4, respectively.

2. Answer to Problem 1

First we review an abstract theorem in [5] toward Theorem 1.1. Let X be a complex Hilbert space with inner product (\cdot, \cdot) and norm $\|\cdot\|$. Let $\varphi, \psi : X \rightarrow [0, \infty]$ be proper lower semicontinuous convex functions on X . We assume for simplicity that the subdifferentials $\partial\varphi, \partial\psi$ are single-valued. Then we consider the abstract Cauchy problem in X :

$$(ACP) \quad \begin{cases} \frac{\partial u}{\partial t} + (\lambda + i\alpha)\partial\varphi(u) + (\kappa + i\beta)\partial\psi(u) - \gamma u = 0, \\ u(0) = u_0, \end{cases}$$

where $\lambda, \kappa \in \mathbb{R}_+$, $\alpha, \beta, \gamma \in \mathbb{R}$ are constants. We need the following conditions on φ, ψ :

(A1) The sublevel set $\{u \in D(\varphi); \varphi(u) \leq c\}$ is compact in X for each $c > 0$.

(A2) $\exists p \in [2, \infty)$ such that $\varphi(\zeta u) = |\zeta|^p \varphi(u)$, $u \in D(\varphi)$, $\zeta \in \mathbb{C}$, $\text{Re } \zeta > 0$.

(A3) $\exists q \in [2, \infty)$ such that $\psi(\zeta u) = |\zeta|^q \psi(u)$, $u \in D(\psi)$, $\zeta \in \mathbb{C}$, $\text{Re } \zeta > 0$.

(A4) $\exists c_p \geq 0$ such that for $u, v \in D(\partial\varphi)$ and $\varepsilon > 0$,

$$|\text{Im}(\partial\varphi(u) - \partial\varphi(v), u - v)| \leq c_p \text{Re}(\partial\varphi(u) - \partial\varphi(v), u - v).$$

(A5) $\exists c_q \geq 0$ such that for $u \in D(\partial\varphi)$ and $\varepsilon > 0$,

$$|\text{Im}(\partial\varphi(u), \partial\psi_\varepsilon(u))| \leq c_q \text{Re}(\partial\varphi(u), \partial\psi_\varepsilon(u)),$$

where $\partial\psi_\varepsilon$ is the Yosida approximation of $\partial\psi$: $\partial\psi_\varepsilon := \varepsilon^{-1}(1 - (1 + \varepsilon\partial\psi)^{-1})$.

The following theorem is established in [5].

Theorem 2.1 ([5, Theorem 4.1]). *Assume that (A1)–(A5) are satisfied. Assume that α/λ and β/κ satisfy*

$$\frac{|\alpha|}{\lambda} \leq c_p^{-1}, \quad \left(\frac{\alpha}{\lambda}, \frac{\beta}{\kappa}\right) \in \text{CGL}(c_q^{-1}).$$

Then for any $u_0 \in D(\varphi) \cap D(\psi)$ there exists a global strong solution $u(\cdot) \in C([0, \infty); X)$ to (ACP) such that

- (a) $u(\cdot) \in C^{0,1/2}([0, T]; X)$, $T > 0$,
- (b) $(du/dt)(\cdot), \partial\varphi(u(\cdot)), \partial\psi(u(\cdot)) \in L^2(0, T; X)$, $T > 0$,
- (c) $\varphi(u(\cdot))$ and $\psi(u(\cdot))$ are absolutely continuous on $[0, T]$ for every $T > 0$,

with the estimates

$$(2.1) \quad \|u(t)\| \leq e^{\gamma t} \|u_0\|, \quad t > 0,$$

$$(2.2) \quad E(u(t)) + \eta \int_0^t (\delta^2 \|\partial\varphi(u(s))\|^2 + \|\partial\psi(u(s))\|^2) ds \leq e^{\gamma+rt} E(u_0), \quad t > 0,$$

where

$$E(u) := \delta^2 \varphi(u) + \psi(u),$$

$\gamma := \max\{\gamma, 0\}$, $r := \max\{p, q\}$ and $\delta, \eta > 0$ are constants.

Next we apply Theorem 2.1 to $(\text{CGL})_{\mathbb{R}^N, \mu}$. In the complex Hilbert space $X := L^2(\mathbb{R}^N)$ we introduce two convex functions on X :

$$(2.3) \quad \varphi(u) := \begin{cases} \frac{1}{2}(\|\nabla u\|_{L^2}^2 + \mu^2 \| |x|u \|_{L^2}^2) & \text{if } u \in D(\varphi) := H^1(\mathbb{R}^N) \cap D(|x|), \\ \infty & \text{otherwise,} \end{cases}$$

$$(2.4) \quad \psi(u) := \begin{cases} \frac{1}{q} \|u\|_{L^q}^q & \text{if } u \in D(\psi) := X \cap L^q(\mathbb{R}^N), \\ \infty & \text{otherwise.} \end{cases}$$

Then their subdifferentials are given by

$$\begin{aligned} \partial\varphi(u) &= -\Delta u + \mu^2 |x|^2 u, \quad u \in D(\partial\varphi) = H^2(\mathbb{R}^N) \cap D(|x|^2), \\ \partial\psi(u) &= |u|^{q-2} u, \quad u \in D(\partial\psi) = X \cap L^{2(q-1)}(\mathbb{R}^N). \end{aligned}$$

To apply Theorem 2.1 with those X , φ and ψ , we prepare some lemmas.

Lemma 2.2. *Let $N \in \mathbb{N}$ and $\mu > 0$. Then for every $u \in H^1(\mathbb{R}^N) \cap D(|x|)$,*

$$(2.5) \quad \|u\|_{L^2}^2 \leq \frac{2}{N} \|\nabla u\|_{L^2} \| |x|u \|_{L^2};$$

in particular,

$$(2.6) \quad N\mu \|u\|_{L^2}^2 \leq \|\nabla u\|_{L^2}^2 + \mu^2 \| |x|u \|_{L^2}^2.$$

Proof. Let $u \in C_0^\infty(\mathbb{R}^N)$ and $\varepsilon > 0$. Let $|x|_\varepsilon := |x|(1 + \varepsilon|x|)^{-1}$ be the Yosida approximation of $|x|$ and $x_\varepsilon := x(1 + \varepsilon|x|)^{-1}$. Then we can obtain

$$(2.7) \quad N \int_{\mathbb{R}^N} \frac{|u(x)|^2}{1 + \varepsilon|x|} dx \leq 2\|\nabla u\|_{L^2} \| |x|_\varepsilon u \|_{L^2} + \varepsilon \|u\|_{L^2} \| |x|_\varepsilon u \|_{L^2}.$$

In fact, observing

$$\begin{aligned} N(1 + \varepsilon|x|)^{-1} &= \operatorname{div} x_\varepsilon + \varepsilon|x|_\varepsilon(1 + \varepsilon|x|)^{-1} \\ &\leq \operatorname{div} x_\varepsilon + \varepsilon|x|_\varepsilon, \end{aligned}$$

we see from integration by parts that

$$\begin{aligned} N \int_{\mathbb{R}^N} \frac{|u(x)|^2}{1 + \varepsilon|x|} dx &\leq \int_{\mathbb{R}^N} (\operatorname{div} x_\varepsilon) |u(x)|^2 dx + \varepsilon \int_{\mathbb{R}^N} |x|_\varepsilon |u(x)|^2 dx \\ &= -2 \int_{\mathbb{R}^N} x_\varepsilon \cdot \operatorname{Re}(u(x) \nabla \overline{u(x)}) dx + \varepsilon \|u\|_{L^2} \| |x|_\varepsilon u \|_{L^2} \\ &\leq 2\|\nabla u\|_{L^2} \| |x|_\varepsilon u \|_{L^2} + \varepsilon \|u\|_{L^2} \| |x|_\varepsilon u \|_{L^2}. \end{aligned}$$

Since $C_0^\infty(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N)$, (2.7) is true also for $u \in H^1(\mathbb{R}^N)$. Letting $\varepsilon \downarrow 0$ in (2.7) for $u \in H^1(\mathbb{R}^N) \cap D(|x|)$, we obtain (2.5). (2.6) is a consequence of (2.5). \square

Lemma 2.3 ([5, Lemma 6.2]). *Let $q \geq 2$. Then for $u \in H^2(\mathbb{R}^N)$ and $\varepsilon > 0$,*

$$(2.8) \quad |\operatorname{Im}(-\Delta u, \partial\psi_\varepsilon(u))_{L^2}| \leq \frac{q-2}{2\sqrt{q-1}} \operatorname{Re}(-\Delta u, \partial\psi_\varepsilon(u)).$$

Lemma 2.4. *Let $V : \mathbb{R}^N \rightarrow \mathbb{R}$ be a nonnegative function. Then for $\varepsilon > 0$ and $u \in L^2(\mathbb{R}^N)$ with $Vu \in L^2(\mathbb{R}^N)$,*

$$(2.9) \quad (Vu, \partial\psi_\varepsilon(u))_{L^2} = \int_{\mathbb{R}^N} V|u_\varepsilon|^q dx + \varepsilon \int_{\mathbb{R}^N} V|u_\varepsilon|^{2(q-1)} dx$$

where $u_\varepsilon := (1 + \varepsilon\partial\psi)^{-1}u$. Consequently, $(Vu, \partial\psi_\varepsilon(u))_{L^2}$ is real and nonnegative.

Proof. Let $\varepsilon > 0$ and $u \in L^2(\mathbb{R}^N)$ with $Vu \in L^2(\mathbb{R}^N)$. Setting $u_\varepsilon := (1 + \varepsilon\partial\psi)^{-1}u$, we see that

$$u = u_\varepsilon + \varepsilon|u_\varepsilon|^{q-2}u_\varepsilon, \quad \partial\psi_\varepsilon(u) = |u_\varepsilon|^{q-2}u_\varepsilon.$$

Substituting these identities into $(Vu, \partial\psi_\varepsilon(u))_{L^2}$, we can obtain (2.9). \square

Lemma 2.5. *Let $q \geq 2$. Then for $u \in D(\partial\varphi)$ and $\varepsilon > 0$,*

$$(2.10) \quad |\operatorname{Im}(\partial\varphi(u), \partial\psi_\varepsilon(u))_{L^2}| \leq \frac{q-2}{2\sqrt{q-1}} \operatorname{Re}(\partial\varphi(u), \partial\psi_\varepsilon(u))_{L^2}.$$

Lemma 2.5 is a consequence of Lemmas 2.3 and 2.4 with $V(x) := \mu^2|x|^2$; note that $\partial\varphi = -\Delta + V(x)$.

Proof of Theorem 1.1. Let $X := L^2(\mathbb{R}^N)$. Let φ and ψ be defined as (2.3) and (2.4). We see from (2.6) that $(-\Delta + \mu^2|x|^2)^{-1}$ is bounded. In fact, (2.6) implies that for every $u \in H^2(\mathbb{R}^N) \cap D(|x|^2)$,

$$\begin{aligned} N\mu\|u\|_{L^2}^2 &\leq \|\nabla u\|_{L^2}^2 + \mu^2\||x|u\|_{L^2}^2 \\ &= ((-\Delta + \mu^2|x|^2)u, u)_{L^2} \\ &\leq \|(-\Delta + \mu^2|x|^2)u\|_{L^2}\|u\|_{L^2}. \end{aligned}$$

Since the potential $|x|^2$ blows up as $|x| \rightarrow \infty$, it follows from [4, Theorem 4.1] that $(-\Delta + \mu^2|x|^2)^{-1}$ is compact in X and hence (A1) is satisfied. (A2) (with $p = 2$) and (A3) are trivial by definition. Since $\partial\varphi$ is nonnegative selfadjoint in X , (A4) is satisfied with $c_p = 0$. Lemma 2.4 implies that (A5) is satisfied with

$$c_q := \frac{q-2}{2\sqrt{q-1}}.$$

Therefore we can apply Theorem 2.1 with those X, φ . Consequently, we obtain the existence part of Theorem 1.1. As in the proof of [5, Theorem 1.1], we can prove (1.3) by virtue of Theorem 2.1 (c). Moreover, (1.4) and (1.5) follow from (2.1) and (2.2), respectively (see Remark 2.1 below). This completes the proof of Theorem 1.1. \square

Remark 2.1. By the definition of φ in (2.3), Theorem 2.1 (b) asserts that

$$u(\cdot), (\Delta - \mu^2|x|^2)u(\cdot) \in L^2(0, T; L^2(\mathbb{R}^N)), \quad T > 0.$$

This fact implies that

$$\Delta u(\cdot), |x|^2 u(\cdot) \in L^2(0, T; L^2(\mathbb{R}^N)), \quad T > 0.$$

This is a direct consequence of the following inequality (see Okazawa [4]):

$$(2.11) \quad \|\Delta u\|_{L^2}^2 + \mu^4\||x|^2 u\|_{L^2}^2 \leq \|(\Delta - \mu^2|x|^2)u\|_{L^2}^2 + 2N\mu^2\|u\|_{L^2}^2, \quad u \in H^2(\mathbb{R}^N) \cap D(|x|^2).$$

3. Answer to Problem 2

In this section we give the proof of Theorem 1.2.

Proof of Theorem 1.2. It suffices to prove (1.7). Let $q < 2^*$. Then $H^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$. Let $u(\cdot)$ and $v(\cdot)$ be the global strong solutions to $(\text{CGL})_{\mathbb{R}^N, \mu}$ with initial data $u_0, v_0 \in H^1(\mathbb{R}^N) \cap D(|x|)$, respectively. Then $w(\cdot) := u(\cdot) - v(\cdot)$ satisfies

$$(3.1) \quad \frac{\partial w}{\partial t} + (\lambda + i\alpha)(-\Delta + \mu^2|x|^2)w + (\kappa + i\beta)(|u|^{q-2}u - |v|^{q-2}v) = \gamma w.$$

Making the L^2 -inner product of (3.1) with w , we have

$$(3.2) \quad \frac{1}{2} \frac{d}{dt} \|w\|_{L^2}^2 + \lambda(\|\nabla w\|_{L^2}^2 + \mu^2\||x|w\|_{L^2}^2) + I = \gamma\|w\|_{L^2}^2,$$

where

$$I := \operatorname{Re} [(\kappa + i\beta)(|u|^{q-2}u - |v|^{q-2}v, w)_{L^2}].$$

Since $\| |u|^{q-2}u - |v|^{q-2}v \| \leq (q-1)(|u|^{q-2} + |v|^{q-2})|w|$, we have

$$(3.3) \quad \begin{aligned} |I| &\leq (q-1)\sqrt{\kappa^2 + \beta^2} \int_{\mathbb{R}^N} (|u|^{q-2} + |v|^{q-2})|w|^2 dx \\ &\leq (q-1)\sqrt{\kappa^2 + \beta^2} (\|u\|_{L^q}^{q-2} + \|v\|_{L^q}^{q-2}) \|w\|_{L^q}^2, \end{aligned}$$

where we used the Hölder inequality in the second inequality. We see from (1.5) that

$$\|u(t)\|_{L^q}^q \leq qe^{\gamma+qt} E_\mu(u_0), \quad \|v(t)\|_{L^q}^q \leq qe^{\gamma+qt} E_\mu(v_0).$$

Hence we have

$$(3.4) \quad \|u(t)\|_{L^q}^{q-2} + \|v(t)\|_{L^q}^{q-2} \leq K_1 e^{\gamma+(q-2)t},$$

where

$$K_1 := q^{1-2/q} \left[E_\mu(u_0)^{1-2/q} + E_\mu(v_0)^{1-2/q} \right].$$

On the other hand, we use the Gagliardo-Nirenberg inequality

$$(3.5) \quad \|w\|_{L^q} \leq C \|w\|_{L^2}^{1-a} \|\nabla w\|_{L^2}^a,$$

where $a := N(1/2 - 1/q) \in [0, 1)$ and $C = C(q, N)$ is a positive constant. Applying (3.4) and (3.5) to (3.3), we see by the Young inequality that

$$\begin{aligned} |I| &\leq (q-1)\sqrt{\kappa^2 + \beta^2} C K_1 e^{\gamma+(q-2)t} \|w\|_{L^2}^{2(1-a)} \|\nabla w\|_{L^2}^{2a} \\ &\leq K_2 e^{\frac{\gamma+(q-2)}{1-a}t} \|w\|_{L^2}^2 + \frac{\lambda}{2} \|\nabla w\|_{L^2}^2, \end{aligned}$$

where

$$K_2 := \left(\frac{2}{\lambda} \right)^{a/(1-a)} \left[(q-1)\sqrt{\kappa^2 + \beta^2} C K_1 \right]^{1/(1-a)}.$$

Plugging this inequality with (3.2), we obtain

$$(3.6) \quad \frac{d}{dt} \|w\|_{L^2}^2 + \lambda (\|\nabla w\|_{L^2}^2 + \mu^2 \| |x|w \|_{L^2}^2) \leq 2 \left(\gamma + K_2 e^{\frac{\gamma+(q-2)}{1-a}t} \right) \|w\|_{L^2}^2.$$

Setting

$$K(t) := 2 \left(\gamma + K_2 e^{\frac{\gamma+(q-2)}{1-a}t} \right),$$

we have

$$\frac{d}{ds} \left[e^{-\int_0^s K(r) dr} \|w(s)\|_{L^2}^2 \right] + \lambda e^{-\int_0^s K(r) dr} (\|\nabla w(s)\|_{L^2}^2 + \mu^2 \| |x|w(s) \|_{L^2}^2) \leq 0.$$

Integrating this inequality on $[0, t]$ for $t > 0$, we obtain (1.7). \square

4. Answer to Problem 3

Let $u_\mu(\cdot)$ be the unique global strong solution to $(\text{CGL})_{\mathbb{R}^N, \mu}$ ($\mu > 0$) constructed in Theorems 1.1 and 1.2. To prove Theorem 1.3 we need a priori estimate of $\| |x| u_\mu(\cdot) \|_{L^2}$ independent of μ .

Lemma 4.1. *Let $N, \lambda + i\alpha, \kappa + i\beta, \gamma, \mu$ be the same as in Theorem 1.2. Let $u_\mu(\cdot)$ be the solution to $(\text{CGL})_{\mathbb{R}^N, \mu}$ with $u_\mu(0) = u_0 \in H^1(\mathbb{R}^N) \cap D(|x|^2)$. Then for every $t > 0$,*

$$(4.1) \quad \| |x|^2 u_\mu(t) \|_{L^2} \leq e^{\gamma t} \left(ct \|u_0\|_{L^2} + \| |x|^2 u_0 \|_{L^2} \right),$$

where $c > 0$ is a constant depending only on $\lambda + i\alpha$.

Proof. We give a formal proof. The proof can be justified by using the Yosida approximation of $|x|^2$. Making the inner product of the equation in $(\text{CGL})_{\mathbb{R}^N, \mu}$ with $|x|^4 u_\mu(\cdot)$, we have

$$(4.2) \quad \frac{1}{2} \frac{d}{dt} \| |x|^2 u_\mu \|_{L^2}^2 + J - \gamma \| |x|^2 u_\mu \|_{L^2}^2 \leq 0,$$

where

$$J := \text{Re} [(\lambda + i\alpha)(-\Delta u_\mu + \mu^2 |x|^2 u_\mu, |x|^4 u_\mu)_{L^2}].$$

Applying integration by parts and the Schwarz inequality, we obtain

$$(4.3) \quad \begin{aligned} J &\geq \lambda \| |x|^2 \nabla u_\mu \|_{L^2}^2 - 4\sqrt{\lambda^2 + \alpha^2} \| |x|^2 \nabla u_\mu \|_{L^2} \| |x| u_\mu \|_{L^2} \\ &\geq -c \| |x| u_\mu \|_{L^2}^2, \end{aligned}$$

where $c := (4/\lambda)(\lambda^2 + \alpha^2)$. On the other hand, it follows from the Schwarz inequality and (1.4) that

$$\| |x| u_\mu(t) \|_{L^2}^2 \leq e^{\gamma t} \|u_0\|_{L^2} \| |x|^2 u_\mu(t) \|_{L^2}.$$

Applying this inequality to (4.3), we see from (4.2) that

$$\frac{1}{2} \frac{d}{dt} \| |x|^2 u_\mu(t) \|_{L^2}^2 - ce^{\gamma t} \|u_0\|_{L^2} \| |x|^2 u_\mu(t) \|_{L^2} - \gamma \| |x|^2 u_\mu(t) \|_{L^2}^2 \leq 0,$$

which implies that

$$\frac{d}{dt} \left(e^{-\gamma t} \| |x|^2 u_\mu(t) \|_{L^2} \right) \leq c \|u_0\|_{L^2}.$$

Integrating this inequality on $[0, t]$ yields (4.1). \square

Now we are in position to complete the proof of Theorem 1.3 which answers to **Problem 3**.

Proof of Theorem 1.3. Let $u_\mu(\cdot)$ be the unique global strong solution to $(\text{CGL})_{\mathbb{R}^N, \mu}$ with $u_\mu(0) = u_0 \in H^1(\mathbb{R}^N) \cap D(|x|^2)$. Set $w_{\mu, \nu}(\cdot) := u_\mu(\cdot) - u_\nu(\cdot)$ for $\mu, \nu \in (0, 1]$. Similarly in deriving (3.6), we have

$$\frac{1}{2} \frac{d}{dt} \|w_{\mu, \nu}\|_{L^2}^2 + \frac{\lambda}{2} \|\nabla w_{\mu, \nu}\|_{L^2}^2 + I_{\mu, \nu} \leq \frac{K(t)}{2} \|w_{\mu, \nu}\|_{L^2}^2,$$

where

$$\begin{aligned} I_{\mu,\nu} &:= \operatorname{Re} [(\lambda + i\alpha)(\mu^2|x|^2u_\mu - \nu^2|x|^2u_\nu, w_{\mu,\nu})_{L^2}] \\ &= \lambda\mu^2\| |x|w_{\mu,\nu} \|_{L^2}^2 + (\mu^2 - \nu^2)\operatorname{Re} [(\lambda + i\alpha)(|x|^2u_\nu, w_{\mu,\nu})_{L^2}], \end{aligned}$$

and $K(\cdot)$ is the same function as in Theorem 1.2. From (4.1) we have

$$\begin{aligned} I_{\mu,\nu} &\geq -\sqrt{\lambda^2 + \alpha^2}|\mu^2 - \nu^2|\| |x|u_\nu \|_{L^2}\|w_{\mu,\nu}\|_{L^2} \\ &\geq -M(t)|\mu^2 - \nu^2|\|w_{\mu,\nu}\|_{L^2}, \end{aligned}$$

where

$$M(t) := \sqrt{\lambda^2 + \alpha^2}e^{\gamma t} \left(ct\|u_0\|_{L^2} + \| |x|^2u_0 \|_{L^2} \right).$$

Hence we obtain

$$(4.4) \quad \frac{d}{dt}\|w_{\mu,\nu}\|_{L^2} \leq \frac{K(t)}{2}\|w_{\mu,\nu}\|_{L^2} + M(t)|\mu^2 - \nu^2|.$$

Applying the Gronwall lemma to (4.4) yields

$$\|w_{\mu,\nu}(t)\|_{L^2} \leq |\mu^2 - \nu^2| \int_0^t e^{\int_s^t \frac{K(r)}{2} dr} M(s) ds.$$

This inequality implies that for every $T > 0$,

$$\sup_{0 < t < T} \|w_{\mu,\nu}(t)\|_{L^2} \leq |\mu^2 - \nu^2| \int_0^T e^{\int_s^T \frac{K(r)}{2} dr} M(s) ds.$$

This implies that $\{u_\mu(\cdot)\}$ satisfies the Cauchy condition in $C([0, T]; L^2(\mathbb{R}^N))$ and hence there exists $u \in C([0, \infty); L^2(\mathbb{R}^N))$ such that

$$u_\mu(\cdot) \rightarrow u(\cdot) \quad (\mu \downarrow 0) \quad \text{strongly in } C([0, T]; L^2(\mathbb{R}^N)).$$

We see from (1.4), (1.5) and (2.11) that

$$\{\Delta u_\mu(\cdot)\} \text{ and } \{|u_\mu|^{q-2}u_\mu(\cdot)\} \text{ are bounded in } L^2(0, T; L^2(\mathbb{R}^N)).$$

Moreover, (4.1) implies that

$$\{|x|^2u_\mu(\cdot)\} \text{ is also bounded in } L^2(0, T; L^2(\mathbb{R}^N)).$$

Since Δ , $|x|^2$ and $\partial/\partial t$ are weakly closed as operators in $L^2(0, T; L^2(\mathbb{R}^N))$, it follows that $\Delta u(\cdot)$, $|x|^2u(\cdot)$, $(\partial u/\partial t)(\cdot) \in L^2(0, T; L^2(\mathbb{R}^N))$ and

$$\begin{aligned} \Delta u_\mu(\cdot) &\rightarrow \Delta u(\cdot) \quad \text{weakly in } L^2(0, T; L^2(\mathbb{R}^N)), \\ \mu^2|x|^2u_\mu(\cdot) &\rightarrow 0 \quad \text{weakly in } L^2(0, T; L^2(\mathbb{R}^N)), \\ (\partial u_\mu/\partial t)(\cdot) &\rightarrow (\partial u/\partial t)(\cdot) \quad \text{weakly in } L^2(0, T; L^2(\mathbb{R}^N)). \end{aligned}$$

We can also see from the demiclosedness of $\partial\psi$ as operators in $L^2(0, T; L^2(\mathbb{R}^N))$ that $|u|^{q-2}u(\cdot) \in L^2(0, T; L^2(\mathbb{R}^N))$ and

$$|u_\mu|^{q-2}u_\mu(\cdot) \rightarrow |u|^{q-2}u(\cdot) \quad \text{weakly in } L^2(0, T; L^2(\mathbb{R}^N)).$$

Therefore $u(\cdot)$ is a global strong solution to $(\text{CGL})_{\mathbb{R}^N, 0}$. □

5. Concluding remarks

We have proved the existence of global strong solutions to $(\text{CGL})_{\mathbb{R}^N, 0}$ under the conditions that

$$\begin{aligned} \left(\frac{\alpha}{\lambda}, \frac{\beta}{\kappa}\right) &\in \text{CGL}(c_q^{-1}), \\ 2 &\leq q < 2^*, \\ u_0 &\in H^1(\mathbb{R}^N) \cap D(|x|^2). \end{aligned}$$

There are two comments; one is about the initial data u_0 and the other is about the exponent q .

(I) If $u_0 \in H^1(\mathbb{R}^N)$, then we can approximate u_0 by

$$u_{0,n} := (1 + n^{-1}|x|^2)^{-1}u_0.$$

As in the proof of Theorem 1.3 we can see that the corresponding solution $u_n(\cdot)$ with $u_n(0) = u_{0,n}$ converges to the desired solution.

(II) For the uniqueness we assumed that $2 \leq q < 2^*$; and hence we obtain the solution to $(\text{CGL})_{\mathbb{R}^N, 0}$ for such exponent q . On the other hand, Ginibre-Velo [2] have already proved the existence of solutions to $(\text{CGL})_{\mathbb{R}^N, 0}$ under the mild condition that “ $2 \leq q < \infty$ ”. The key of their proof lies in the compactness of $H^1(\Omega) \hookrightarrow L^2(\Omega)$ for a bounded domain $\Omega \subset \mathbb{R}^N$. Our method lies in another compactness $H^1(\mathbb{R}^N) \cap D(|x|) \hookrightarrow L^2(\mathbb{R}^N)$. In the future we shall improve our method by using the compactness $H^1(\mathbb{R}^N) \cap D(V) \hookrightarrow L^2(\mathbb{R}^N)$, where $V : \mathbb{R}^N \rightarrow \mathbb{R}$ is a nonnegative function satisfying

$$\lim_{|x| \rightarrow \infty} V(x) = \infty.$$

Choosing V properly, we would show the existence under the condition that $2 \leq q < \infty$.

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